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YEAR 11 MATHEMATICS SPECIALIST

TEST 6, 2018

(Proofs and Complex Numbers)

Section One: Calculator Free

Student's Name: Solutions.

Total Marks: 45
Time Allowed: 45 mins

MATERIAL REQUIRED/RECOMMENDED FOR THIS TEST

Standard Items: Pens, pencils, eraser, ruler

Special Items: WACE Formula Sheet

INSTRUCTIONS TO STUDENTS

Do not open this paper until instructed to do so.
You are required to answer ALL questions.
Write answers in the spaces provided beneath each question.
Marks are shown with the questions.

Show all working clearly, in sufficient detail to allow your answers to be checked and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks.

It is recommended that students **do not use pencil**, except in diagrams.

1. [6 marks]

Plot the following complex numbers on the Argand diagram.

a) $z_1 = 2+3i$

b) $z_2 = -3-4i$

c) $z_3 = \overline{z_1}$

d) $z_4 = z_1 i$

$= 2i - 3$

e) $z_5 = |z_2|$

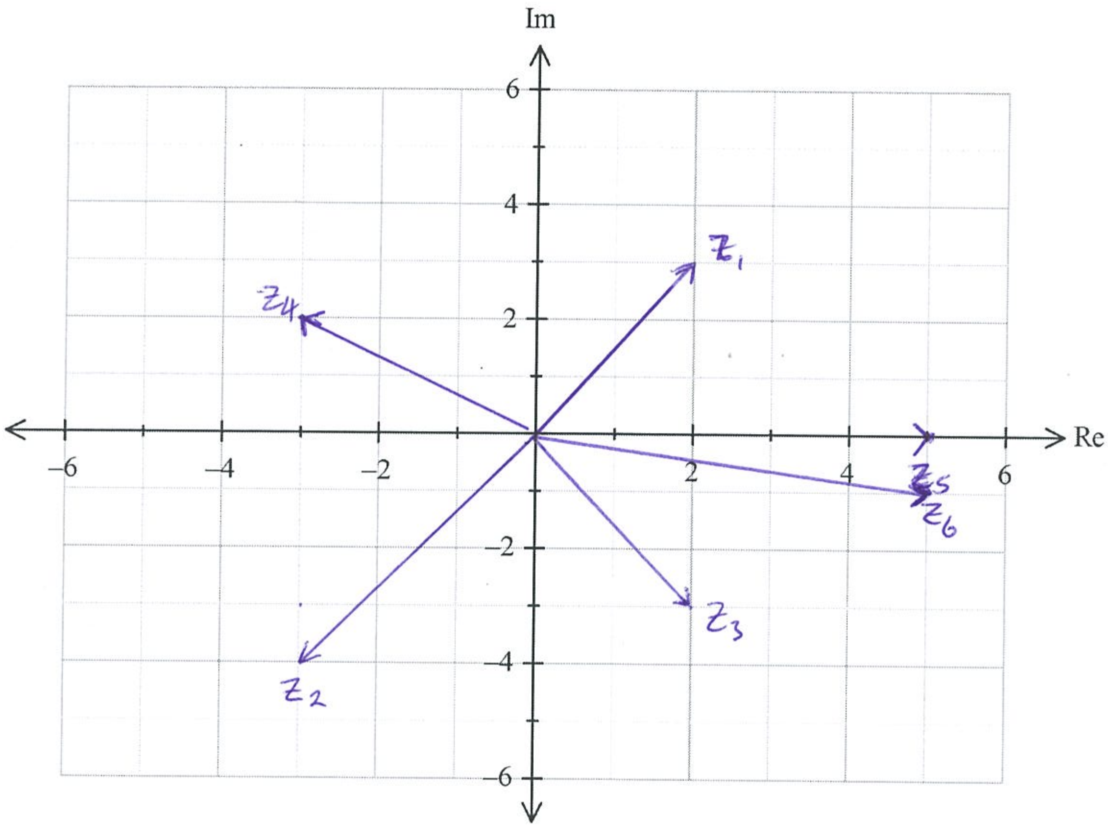
$= \sqrt{3^2+4^2}$

$= 5$

f) $z_6 = z_1 - \overline{z_2}$

$2+3i-(-3+4i)$

$= 5-i$



6

2. [3, 5 = 8 marks]

a) Prove that the square of an odd number add 11 is a multiple of 4.

RTP $(2K+1)^2 + 11 = 4A$ where $A \in \mathbb{Z}$

LHS $4K^2 + 4K + 1 + 11$

$= 4K^2 + 4K + 12$

$= 4(K^2 + K + 3)$

$= 4A$

$\therefore$ multiple of 4.

b) Prove that the product of 3 consecutive even numbers is divisible by 24.

RTP $(2K)(2K+2)(2K+4) = 24A$ where $A \in \mathbb{Z}$

LHS $= 2K(4K^2 + 12K + 8)$

$= 8K(K^2 + 3K + 2)$

$= 8K(K+1)(K+2)$

As K, $K+1$, $K+2$ are 3 consecutive numbers one of them will be a multiple of 3

ie $K(K+1)(K+2) = 3M$

$\therefore$ LHS $= 8 \times 3M$

$= 24M$

$\therefore$ multiple of 24.

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3. [8 marks]

Given that $z = 2 + 3i$ and $w = 4 - i$ determine:

a) zw

$= (2+3i)(4-i)$

$= 8 - 2i + 12i + 3$

$= 11 + 10i$

b) $\bar{z} \times \bar{w}$

$= (2-3i)(4+i)$

$= 8 + 2i - 12i + 3$

$= 11 - 10i$

or $\bar{z} \times \bar{w} = \overline{zw}$

$= 11 - 10i$

c) $\frac{w}{z}$

$\frac{(4-i)}{(2+3i)} \times \frac{(2-3i)}{(2-3i)}$

$= \frac{8 - 12i - 2i - 3}{4+9}$

$= \frac{5 - 14i}{13}$

d) $Re(z) + Im(w)$

$= 2 + -1$

$= 1$

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4. [7 marks]

Prove by mathematical induction that, for $n \in \mathbb{Z}^+$,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

1. For $n = 1$: $LHS = 1$ $\quad RHS = 4 - \frac{1+2}{2^0} = 4 - 3 = 1$

$\therefore$ true for $n = 1$ ✓

2. Assume true for $n = K$

i.e. $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + K\left(\frac{1}{2}\right)^{K-1} = 4 - \frac{K+2}{2^{K-1}}$ ✓

3 Consider $n = K + 1$

RTP $1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \ldots + K\left(\frac{1}{2}\right)^{K-1} + (K+1)\left(\frac{1}{2}\right)^K = 4 - \frac{K+3}{2^K}$ ✓

$$LHS = 4 - \frac{K+2}{2^{K-1}} + (K+1)\left(\frac{1}{2}\right)^K$$

$$= 4 - \frac{K+2}{2^{K-1}} \times \left(\frac{2}{2}\right) + \frac{K+1}{2^K}$$ ✓

$$= 4 - \frac{2K+4}{2^K} + \frac{K+1}{2^K}$$ ✓

$$= 4 - \left(\frac{2K+4}{2^K} - \frac{K+1}{2^K}\right)$$

$$= 4 - \left(\frac{K+3}{2^K}\right)$$ ✓

$= RHS$.

4. As the statement is true for $n = 1$ and if true for $n = K$ then it is true for $n = K+1$ by mathematical induction the statement is true $\forall n \in \mathbb{Z}^+$. ✓

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5. [2, 4 = 6 marks]

Consider the complex numbers $z = 1 + 2\mathrm{i}$ and $w = 2 + a\mathrm{i}$, where $a \in \mathbb{R}$. Find a when:

(a) $|w| = 2|z|$;

$\sqrt{2^2+a^2} = 2\sqrt{1^2+2^2}$ ✓

$4+a^2 = 4(5)$

$= 20$

$a^2 = 16$

$a = \pm 4$ ✓

(b) Re $(zw) = 2$ Im(zw).

$zw = (1+2i)(2+ai)$

$= 2 + ai + 4i - 2a$ ✓

$\equiv 2-2a + (a+4)i$

$\left.\begin{array}{l} R(zw) = 2-2a \\ \mathrm{Im}(zw) = a+4 \end{array}\right\}$ ✓

$2-2a = 2(a+4)$ ✓

$= 2a+8$

$-6 = 4a$

$a = \frac{-6}{4}$ ✓

$a = -\frac{3}{2}$

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6. **[5 marks]**

Use the method of proof by contradiction to prove that $\sqrt{6}$ is irrational.

Assume $\sqrt{6}$ is rational and can be expressed as $\frac{a}{b}$ where a & b have no common factors

$$\frac{a}{b} = \sqrt{6} \Rightarrow \frac{a^2}{b^2} = 6 \Rightarrow a^2 = 6b^2$$

$$\therefore a^2 = 2(3b^2)$$

$\therefore a^2$ is even

$\Rightarrow$ a is even

let $a = 2K$

$$\frac{a^2}{b^2} = \frac{(2K)^2}{b^2} = 6$$

$$4K^2 = 6b^2$$

$$3b^2 = 2K^2$$

$\therefore 3b^2$ is even

hence b^2 must be even

$\therefore$ b is even.

Since a and b are both even they must have a common factor of 2, which contradicts $\sqrt{6}$ being rational.